• Boolean Algebra

→ Boolean algebra is an algebraic structure defined by a set of elements B, together with two binary operators, + and •, provided that the following Huntington postulates are satisfied:

1. a) The structure is closed with respect to the operator +.
   b) The structure is closed with respect to the operator •.
2. a) The element 0 is an identity element with respect to +, i.e., $x+0=0+x=x$.
   b) The element 1 is an identity element with respect to •, i.e., $x \cdot 1 = 1 \cdot x = x$.
3. a) The structure is commutative with respect to +, i.e., $x+y=y+x$.
   b) The structure is commutative with respect to •, i.e., $x \cdot y = y \cdot x$.
4. a) The operator • is distributive over +, i.e., $x \cdot (y+z) = (x \cdot y) + (x \cdot z)$.
   b) The operator + is distributive over •, i.e., $x + (y \cdot z) = (x+y) \cdot (x+z)$.
5. For every element $x \in B$, there exists an element $x' \in B$ (called the complement of x) such that a) $x+x'=1$ and b) $x \cdot x' = 0$.
6. There exist at least two elements $x, y \in B$ such that $x \neq y$.

→ Boolean algebra resembles ordinary algebra in some respects.

→ To realize Boolean algebra, one must show

1. the elements of the set B,
2. the rules of operation for two binary operators, and
3. the set of elements, B, together with the two operators, satisfy the six Huntington postulates.

→ We deal only with a two-valued Boolean algebra (i.e., a Boolean algebra with only two elements).

→ Two-valued Boolean algebra has applications in set theory (the algebra of classes) and in propositional logic.
→ The application of Boolean algebra to gate-type circuits commonly used in digital devices and computers is our interest.

Two-Valued Boolean Algebra

→ A two-valued Boolean algebra is defined on a set of two elements, $B = \{0, 1\}$, with rules for the two binary operators + and · as shown in the following operator tables:

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

x	x'
0	1
1	0

→ These rules are exactly same as the AND, OR and NOT operations, respectively.
→ The Huntington postulates are valid for the set $B = \{0, 1\}$ and the two binary operators + and ·.

- Basic Theorems and Properties of Boolean Algebra

Duality

→ The important property of Boolean algebra is duality principle and states that every algebraic expression deducible from the postulates of Boolean algebra remains valid if the operators and identity elements are interchanged.
→ The duality principle has many applications.
→ If the dual of an algebraic expression is desired, we simply interchange OR and AND operations and replace 1's by 0's and 0's by 1's

Basic Theorems

→ The theorems, like the postulates are listed in pairs, each relation is the dual of the one paired with it.
→ The postulates are basic axioms of the algebraic structure and need no proof.

Postulate 2	(a)	$x + 0 = x$	(b)	$x \cdot 1 = x$
Postulate 5	(a)	$x + x' = 1$	(b)	$x \cdot x' = 0$
Theorem 1	(a)	$x + x = x$	(b)	$x \cdot x = x$
Theorem 2	(a)	$x + 1 = 1$	(b)	$x \cdot 0 = 0$
Theorem 3, involution		$(x')' = x$		
Postulate 3, commutative	(a)	$x + y = y + x$	(b)	$xy = yx$
Theorem 4, associative	(a)	$x + (y+z) = (x+y)+z$	(b)	$x(yz) = (xy)z$
Postulate 4, distributive	(a)	$x(y+z) = xy + xz$	(b)	$x + yz = (x+y)(x+z)$
Theorem 5, De Morgan	(a)	$(x+y)' = x'y'$	(b)	$(xy)' = x' + y'$
Theorem 6, absorption	(a)	$x + xy = x$	(b)	$x(x+y) = x$

→ The theorems must be proven from the postulates.

Theorem 1(a): $x + x = x$

Statement	Justification
$x + x = (x+x) \cdot 1$	Postulate 2(b)
$= (x+x)(x+x')$	5(a)
$= x + xx'$	4(b)
$= x + 0$	5(b)
$= x$	2(a)

Theorem 1(b): $x \cdot x = x$

Statement	Justification
$x \cdot x = xx + 0$	Postulate 2(a)
$= xx + xx'$	5(b)
$= x(x+x')$	4(a)
$= x \cdot 1$	5(a)
$= x$	2(b)

→ Any dual theorem can be similarly derived from the proof of its corresponding theorem.

Theorem 2(a): $x + 1 = 1$

Statement	Justification
$x + 1 = 1 \cdot (x+1)$	Postulate 2(b)
$= (x+x') \cdot (x+1)$	5(a)
$= x + x' \cdot 1$	4(b)
$= x + x'$	2(b)
$= 1$	5(a)

Theorem 2 (b): $x \cdot 0 = 0$ by duality

Theorem 3: $(x')' = x$

→ From postulate 5, we have $x + x' = 1$ and $x \cdot x' = 0$ which together define the complement of x.

→ The complement of x' is x and is also $(x')'$.

→ Since the complement is unique, we have $(x')' = x$.

Theorem 6 (a): $x + xy = x$

	Statement	Justification
$x + xy$	$= x \cdot 1 + xy$	Postulate 2 (b)
	$= x(1+y)$	4 (a)
	$= x(y+1)$	3 (a)
	$= x \cdot 1$	2 (a)
	$= x$	2 (b)

Theorem 6(b): $x(x+y)$ by duality.

→ The theorems of Boolean algebra can be proven by means of truth tables.

→ In truth tables, both sides of the relation are checked to see whether they yield identical results for all possible combinations of the variables involved.

→ The following truth table verifies the first absorption theorem, $x + xy = x$:

x	y	xy	$x + xy$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

→ The following truth table verifies the first DeMorgan's theorem, $(x+y)' = x'y'$:

x	y	$x+y$	$(x+y)'$	x'	y'	$x'y'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Operator Precedence

→ The operator precedence for evaluating Boolean expressions is (1) parentheses, (2) NOT, (3) AND, and (4) OR.

- → Consider the truth table for first De Morgan's theorem, the left side of the expression is $(x+y)'$
- → Therefore, the expression inside the parentheses is evaluated first and the result is then complemented.
- → The right side of the expression is $x'y'$, so the complement of x and the complement of y are both evaluated first and the result is then ANDed.