Propositional Logic– Solution

1) Translate the following Propositional Logic to English sentences.
Let:

- E=Liron is eating
- H=Liron is hungry

(a) $E \Rightarrow \neg H$
Answer: If Liron is eating, then Liron is not hungry

(b) $E \wedge \neg H$
Answer: Liron is eating and not hungry

(c) $\neg(H \Rightarrow \neg E)$
Answer: Liron is hungry and eating

2) Translate the following English sentences to Propositional Logic.
Propositions: (R)aining, Liron is (S)ick, Liron is (H)ungry, Liron is (HA)appy, Liron owns a (C)at, Liron owns a (D)og

(a) It is raining if and only if Liron is sick
Answer: $R \Leftrightarrow S$

(b) If Liron is sick then it is raining, and vice versa
Answer: $(S \Rightarrow R) \wedge (R \Rightarrow S)$ (which is equivalent to $R \Leftrightarrow S$)

(c) It is raining is equivalent to Liron is sick
Answer: $R \Leftrightarrow S$

(d) Liron is hungry but happy
Answer: $H \wedge HA$

(e) Liron either owns a cat or a dog
Answer: $(C \wedge \neg D) \vee (\neg C \wedge D)$

3) Which of the following propositions are tautologies? Which are contradictions? Why?

(a) Three is a prime number.
Answer: neither a tautology nor a contradiction

(b) It is raining or it is not raining.
Answer: tautology

(c) It is raining (P) and it is not raining ($\neg P$).
Answer: contradiction

Example reasoning:
All rows in the truth table evaluate to false.

P	$P \wedge \neg P$
t	f
f	f

4) Which of the following propositions are tautologies? Why?

(a) P
Answer: not a tautology

(b) $P \Rightarrow P$
Answer: tautology

(c) $(P \Rightarrow P) \Rightarrow P$
Answer: not a tautology
Example reasoning:
Not all rows in the truth table evaluate to true.

P	P ⇒ P	(P ⇒ P) ⇒ P
t	t	t
f	t	f

(d) $P \Rightarrow (P \Rightarrow P)$
Answer: tautology

5) Which of the two following propositions are equivalent in the sense that one can always be substituted for the other one in any proposition without changing its truth value? Why?

(a) first proposition:$P \Rightarrow Q$ second proposition: $\neg P \vee Q$
Answer: yes
Example reasoning:
All rows in the truth table evaluate to the same truth value.

P	Q	P ⇒ Q	¬ P ∨ Q
t	t	t	t
t	f	f	f
f	t	t	t
f	f	t	t

(b) first proposition: $\neg P$ second proposition: $P \Rightarrow False$
Answer: yes

(c) first proposition: $\neg P$ second proposition: $False \Rightarrow P$
Answer: no

(d) first proposition: $\neg P$ second proposition: $\neg P \vee Q$
Answer: no

6) Is it possible that

(a) $(KB \models S)$ and $(\neg KB \models S)$
Answer: Yes. For example, if $S \equiv TRUE$, then any interpretation that satisfies KB or $\neg KB$ also satisfies S.

(b) $(KB \models S)$ and $(KB \models \neg S)$
Answer: Yes. For example, if $KB \equiv \mathit{FALSE}$, then KB entails any sentence, including S and $\neg S$.

(c) $(KB \models S)$ and $(KB \not\models S)$
Answer: No. Either all the interpretations that satisfy KB also satisfy S $(KB \models S)$, or there is an interpretation that satisfies KB but not S $(KB \not\models S)$. Both cannot be true at the same time.

(d) $(KB \models S)$ and $(KB \not\models \neg S)$
Answer: Yes. For example, if $KB \equiv \mathit{TRUE}$ and $S \equiv \mathit{TRUE}$, then $\mathit{TRUE} \models \mathit{TRUE}$ and $\mathit{TRUE} \not\models \mathit{FALSE}$.

(e) $(KB \not\models S)$ and $(KB \not\models \neg S)$
Answer: Yes. For example, if $KB \equiv \mathit{TRUE}$, then it cannot entail a sentence S unless S is a tautology. So, if we pick $S \equiv P$, where P is a propositional symbol, then $\mathit{TRUE} \not\models P$ and $\mathit{TRUE} \not\models \neg P$.

(f) $(KB \not\models S)$ and $(\neg KB \not\models S)$
Answer: Yes. For example, if $KB \equiv P$ and $S \equiv Q$, where P and Q are propositional symbols, then $P \not\models Q$ and $\neg P \not\models Q$.

If so, provide an example. If not, explain why it is impossible.

7) Prove that $P \wedge Q \models P \vee Q$.

Answer:

P	Q	$P \wedge Q$	$P \vee Q$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	f

Since every interpretation that satisfies $P \wedge Q$ also satisfies $P \vee Q$, it holds that $P \wedge Q \models P \vee Q$.

8) Consider the following popular puzzle. When asked for the ages of her three children, Mrs. Baker says that Alice is her youngest child if Bill is not her youngest child, and that Alice is not her youngest child if Carl is not her youngest child. Write down a knowledge base that describes this riddle and the necessary background knowledge that only one of the three children can be her youngest child. Show with resolution that Bill is her youngest child.

Answer:
Let the propositions A, B and C denote that Mrs. Baker's youngest child is Alice, Bill and Carl, respectively. We have the following clauses for the background knowledge:

1 $A \vee B \vee C$ (One child has to be the youngest.)

2 $\neg A \vee \neg B$ (Alice and Bill cannot both be the youngest.)

3 $\neg A \vee \neg C$

4 $\neg B \vee \neg C$

The following clauses represent the information from Mrs. Baker:

5 $B \vee A$ (Alice is her youngest child if Bill is not her youngest child. That is, $\neg B \Rightarrow A$.)

6 $C \vee \neg A$ (Alice is not her youngest child if Carl is not her youngest child. That is, $\neg C \Rightarrow \neg A$.)

We want to show that Bill is the youngest child. Negating this, we get the following clause:

7 $\neg B$ (Assume that Bill is not the youngest child.)

We use resolution to derive the empty clause as follows:

8 (from 5,7) A

9 (from 3,6) $\neg A$

10 (from 8,9) $\perp$

9) Consider the following popular puzzle. A boy and a girl are talking. "I am a boy" said the child with black hair. "I am a girl" said the child with white hair. At least one of them is lying. Write down a knowledge base that describes this riddle. Show with resolution that both of them are lying.

Answer:

We use the following propositions:

- W_t: White haired child is telling the truth.
- W_b: White haired child is a boy.
- B_t: Black haired child is telling the truth.
- B_b: Black haired child is a boy.

We have the following clauses;

1 $B_b \vee W_b$ ("A boy and a girl are talking" means that at least one of them has to be a boy.)

2 $\neg B_b \vee \neg W_b$ (With the same logic, at least one of them has to be a girl.)

3 $\neg B_t \vee B_b$ (If the black haired child is telling the truth, it has to be a boy. That is, $B_t \Rightarrow B_b$.)

4 $B_t \vee \neg B_b$ (If the black haired child is lying, it has to be a girl. That is, $\neg B_t \Rightarrow \neg B_b$.)

5 $\neg W_t \vee \neg W_b$ (If the white haired child is telling the truth, it has to be a girl. That is, $W_t \Rightarrow \neg W_b$.)

6 $W_t \vee W_b$ (If the white haired child lying, it has to be a boy. That is, $\neg W_t \Rightarrow W_b$.)

7 $\neg B_t \vee \neg W_t$ (At least one of them is lying.)

We want to show that both of them are lying. That is, $\neg B_t \wedge \neg W_t$. Negating this, we get the following clause:

8 $B_t \vee W_t$ (Assume that at least one of them is telling the truth.)

We use resolution to derive the empty clause as follows:

9 (from 3,8) $B_b \vee W_t$

10 (from 5,9) $B_b \vee \neg W_b$

11 (from 1,10) B_b

12 (from 2,10) $\neg W_b$

13 (from 4,11) B_t

14 (from 6,12) W_t

15 (from 7,13) $\neg W_t$

16 (from 14,15) $\bot$

10) In the back of a magazine you find a riddle: "Suppose that liars always speak what is false, and truth-tellers always speak what is true. Further suppose that Amy is either a liar or a truth-teller." The riddle then provides some additional facts about Amy and asks whether Amy has to be a truth-teller. Excitedly, you encode the facts in propositional logic and implement a resolution procedure on your computer. Since you do not make any mistakes, the computer will give you the correct answer. You ask the computer whether the facts entail that Amy is a truth-teller.

a) The computer tells you that the facts entail that Amy is a truth-teller. Since the text states that Amy is either a liar or a truth-teller, can you conclude that Amy is not a liar?

Answer: Yes. You know that Amy has to be a truth-teller (if the facts are true). So, she cannot be a liar according to the text.

b) The computer tells you that the facts do not entail that Amy is a truth-teller. Since the text states that Amy is either a liar or a truth-teller, can you conclude that Amy is a liar?

Answer: No. It could indeed be the case that Amy is not a truth-teller and thus a liar (if the facts are true) but you do not know this for sure. It could also be the case that the computer does not have sufficient information to conclude whether Amy is a truth-teller or a liar. If you want to know whether Amy has to be a liar, you need to ask the computer whether the facts entail that Amy is a liar. If the computer tells you that the facts entail that Amy is a liar, then you know that Amy has to be a liar (if the facts are true).